

Eigenvalues of the Wentzell-Laplace Operator and of the Fourth Order Steklov Problems *

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Abstract

We prove a sharp upper bound and a lower bound for the first nonzero eigenvalue of the Wentzell-Laplace operator on compact manifolds with boundary and an isoperimetric inequality for the same eigenvalue in the case where the manifold is a bounded domain in a Euclidean space. We study some fourth order Steklov problems and obtain isoperimetric upper bound for the first eigenvalue of them. We also find all the eigenvalues and eigenfunctions for two kind of fourth order Steklov problems on a Euclidean ball.

1 Introduction and Statement of the Results

Let $n \geq 2$ and (M, \langle, \rangle) be an n -dimensional compact Riemannian manifold with boundary. We denote by $\overline{\Delta}$ and Δ the Laplace-Beltrami operators on M and ∂M , respectively, and consider the eigenvalue problem for Wentzell boundary conditions

$$\begin{cases} \overline{\Delta}u = 0 & \text{in } M, \\ -\beta\Delta u + \partial_\nu u = \lambda u & \text{on } \partial M, \end{cases} \quad (1.1)$$

where β is a given real number and ∂_ν denotes the outward unit normal derivative. When M is a bounded domain in a Euclidean space, the above problem has been studied recently in [7]. A general derivation of Wentzell boundary conditions can be found in [13]. Note that when $\beta = 0$, the problem (1.1) becomes the second order Steklov problem:

$$\begin{cases} \overline{\Delta}u = 0 & \text{in } M, \\ \partial_\nu u = pu & \text{on } \partial M, \end{cases} \quad (1.2)$$

which has a discrete spectrum consisting in a sequence

$$p_0 = 0 < p_1 \leq p_2 \leq \cdots \rightarrow +\infty.$$

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When $\beta \geq 0$, the spectrum of the Laplacian with Wentzell condition consists in an increasing countable sequence of eigenvalues

$$\lambda_{0,\beta} = 0 < \lambda_{1,\beta} \leq \lambda_{2,\beta} \leq \cdots \rightarrow +\infty,$$

with corresponding real orthonormal (in $L^2(\partial M)$) eigenfunctions u_0, u_1, u_2, \dots . We adopt the convention that each eigenvalue is repeated according to its multiplicity. Consider the Hilbert space

$$H(M) = \{u \in H^1(M), \text{Tr}_{\partial M}(u) \in H^1(\partial M)\}, \quad (1.3)$$

where $\text{Tr}_{\partial M}$ is the trace operator. We define on $H(M)$ the two bilinear forms

$$A_\beta(u, v) = \int_M \bar{\nabla} u \cdot \bar{\nabla} v + \beta \int_{\partial M} \nabla u \cdot \nabla v, \quad B(u, v) = \int_{\partial M} uv, \quad (1.4)$$

where $\bar{\nabla}$ and ∇ are the gradient operators on M and ∂M , respectively. Since we assume β is nonnegative, the two bilinear forms are positive and the variational characterization for the k -th eigenvalue is

$$\lambda_{k,\beta} = \min \left\{ \frac{A_\beta(u, u)}{B(u, u)}, u \in H(M), u \neq 0, \int_{\partial M} uu_i = 0, i = 0, \dots, k-1 \right\}. \quad (1.5)$$

When $k = 1$, the minimum is taken over the functions orthogonal to the eigenfunctions associated to $\lambda_{0,\beta} = 0$, i.e., constant functions. It is easy to see that if $\beta > 0$, p_1 is the first non-zero eigenvalue of the Steklov problem (1.2) and η_1 the first non-zero eigenvalue of the Laplacian acting on functions on ∂M , then we have

$$\lambda_{1,\beta} \geq \beta \eta_1 + p_1, \quad (1.6)$$

with equality holding if and only if any eigenfunction f corresponding to $\lambda_{1,\beta}$ is an eigenfunction corresponding to p_1 and $f|_{\partial M}$ is an eigenfunction corresponding to η_1 .

If M is an n -dimensional Euclidean ball of radius R , then $\lambda_{1,\beta}$ has multiplicity n , the corresponding eigenfunctions are the coordinate functions $x_i, i = 1, \dots, n$, and

$$\lambda_{1,\beta} = \frac{(n-1)\beta + R}{R^2}. \quad (1.7)$$

More generally all the eigenfunctions are the spherical harmonics and the eigenvalue associated to spherical harmonics of order l is (Cf. [7])

$$\frac{k(k+n-2)\beta + kR}{R^2}. \quad (1.8)$$

In this paper we obtain a sharp upper bound for the first nontrivial eigenvalue of the problem (1.1).

Theorem 1.1. *Let $(M, \langle \cdot, \cdot \rangle)$ be an n -dimensional compact connected Riemannian manifold with boundary and β a positive constant. Denote by η_1 the first non-zero eigenvalue of the Laplacian acting on functions on ∂M and $\lambda_{1,\beta}$ the first non-zero eigenvalue of the problem (1.1). If the Ricci curvature of M is bounded below by $-\kappa_0$ for some non-negative constant κ_0 and the principle curvatures of ∂M are bounded below by a positive constant c , then we have*

$$\lambda_{1,\beta} \leq \beta\eta_1 + \frac{2\eta_1 + \kappa_0 + \sqrt{(2\eta_1 + \kappa_0)^2 - 4(n-1)\eta_1 c^2}}{2(n-1)c} \quad (1.9)$$

with equality holding if and only if $\kappa_0 = 0$ and M is isometric to an n -dimensional Euclidean ball of radius $\frac{1}{c}$.

For bounded domains Ω in a Euclidean space \mathbf{R}^n , we have an isoperimetric upper bound for $\lambda_{1,\beta}$ which depends only on the volume of Ω , the area of $\partial\Omega$ and the dimension n .

Theorem 1.2. *Let Ω be a bounded domain with smooth boundary in \mathbf{R}^n and β a positive constant. Then the first non-zero eigenvalue $\lambda_{1,\beta}$ of the problem*

$$\begin{cases} \bar{\Delta}u = 0 & \text{in } \Omega, \\ -\beta\Delta u + \partial_\nu u = \lambda u & \text{on } \partial\Omega, \end{cases} \quad (1.10)$$

satisfies

$$\lambda_{1,\beta} \leq \frac{n|\Omega| + \beta(n-1)|\partial\Omega|}{n|\Omega|(|\Omega|\omega_n^{-1})^{1/n}} \quad (1.11)$$

with equality holding if and only if M is isometric to a ball, where $|\Omega|$, $|\partial\Omega|$ and ω_n denote the volume of Ω , the area of $\partial\Omega$ and the volume of a unit ball of \mathbf{R}^n , respectively.

Our next result provides a lower bound for the same eigenvalue.

Theorem 1.3. *Let $(M, \langle \cdot, \cdot \rangle)$ be an n -dimensional compact connected Riemannian manifold and boundary ∂M and β a positive constant. Assume that the principal curvatures of ∂M are bounded below by a positive constant c .*

i) If the Ricci curvature of M is bounded below by $-\kappa$ for some non-negative constant κ , then the first nonzero eigenvalue of the problem (1.2) satisfies

$$p_1 > \frac{c\eta_1}{2\eta_1 + \kappa}, \quad (1.12)$$

where η_1 is the first non-zero eigenvalue of the Laplacian acting on functions on ∂M .

ii) If M has non-negative Ricci curvature, then the first non-zero eigenvalue $\lambda_{1,\beta}$ of the problem (1.1) satisfies

$$\lambda_{1,\beta} > \frac{\left(1 + (n-1)c\beta + \sqrt{(n-1)^2 c^2 \beta^2 + 2(n-1)c\beta}\right) c}{2} \quad (1.13)$$

In the case of $\kappa = 0$, (1.12) has been proved by Escobar in [8]. An interesting question related to Theorem 1.3 is to find the best possible lower bound for $\lambda_{1,\beta}$ and p_1 . Escobar conjectured that when $\kappa = 0$, $p_1 \geq c$ (Cf. [8]). We believe that (1.13) could be improved as

$$\lambda_{1,\beta} \geq (n-1)\beta c^2 + c, \quad (1.14)$$

with equality holding if and only if M is isometric to a Euclidean ball of radius $\frac{1}{c}$ in \mathbf{R}^n .

Now we come to eigenvalues of fourth order Steklov problems. Let Ω be a bounded bounded domain in \mathbf{R}^n and consider the problem

$$\begin{cases} \overline{\Delta}^2 u - \tau \overline{\Delta} u = 0 & \text{in } \Omega, \\ \frac{\partial^2 u}{\partial \nu^2} = 0 & \text{on } \partial\Omega, \\ \tau \frac{\partial u}{\partial \nu} - \operatorname{div}_{\partial\Omega}(\overline{\nabla}^2 u(\nu)) - \frac{\partial \overline{\Delta} u}{\partial \nu} = \lambda u & \text{on } \partial\Omega, \end{cases} \quad (1.15)$$

in the unknowns u (the eigenfunction), λ (the eigenvalue), where $\tau > 0$ is a fixed positive constant, ν denotes the outer unit normal to $\partial\Omega$, $\operatorname{div}_{\partial\Omega}$ denotes the tangential divergence operator, $\overline{\Delta}$ is the Laplacian on \mathbf{R}^n and $\overline{\nabla}^2 u$ the Hessian of u . For $n = 2$, this problem is related to the study of the vibrations of a thin elastic plate with a free frame and mass concentrated at the boundary. The spectrum consists of a diverging sequence of eigenvalues of finite multiplicity

$$0 = \lambda_0(\Omega) < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \dots \leq \lambda_j(\Omega) \leq \dots, \quad (1.16)$$

where we repeat the eigenvalues according to their multiplicity. The problem (1.15) is the analogue for the biharmonic operator of the classical Steklov problem (1.2) for the Laplace operator and has been studied recently in [4]. The first nonzero eigenvalue of (1.15) is usually called the fundamental tone and can be characterized by means of the Rayleigh principle

$$\lambda_1 = \min_{\substack{0 \neq u \in H^2(\Omega) \\ \int_{\partial\Omega} u = 0}} \frac{\int_{\Omega} |\overline{\nabla}^2 u|^2 + \tau |\overline{\nabla} u|^2}{\int_{\partial\Omega} u^2}. \quad (1.17)$$

If Ω is a ball of radius R in \mathbf{R}^n , then $\lambda_1 = \frac{\tau}{R}$ (Cf. [4]). The following result provides a sharp upper bound for the first nonzero eigenvalue of the problem (1.15) on bounded convex domains in \mathbf{R}^n .

Theorem 1.4. *Let Ω be a bounded smooth domain in \mathbf{R}^n and assume that the principle curvatures of $\partial\Omega$ are bounded below by a positive constant c . Let η_1 be the first non-zero eigenvalue of the Laplacian acting on functions on $\partial\Omega$. Then the first nonzero eigenvalue of the problem (1.15) satisfies*

$$\lambda_1 \leq \frac{\eta_1^2 + \tau \left(\eta_1 + \sqrt{\eta_1^2 - (n-1)\eta_1 c^2} \right)}{(n-1)c} - c\eta_1, \quad (1.18)$$

with equality holding if and only if $\kappa = 0$ and Ω is a ball of radius $\frac{1}{c}$.

Another Steklov problem for the bi-harmonic operator we are interested is as follows:

$$\begin{cases} \overline{\Delta}^2 u = 0 & \text{in } \Omega, \\ \partial_\nu u = \partial_\nu(\overline{\Delta} u) + \xi u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.19)$$

This problem describes the deformation u of the linear elastic supported plate Ω under the action of the transversal exterior force $f(x) = 0$, $x \in \Omega$ with Neumann boundary condition $\partial_\nu u|_{\partial\Omega} = 0$ (see, [25] or p. 32 of [24]) and was first discussed by J. R. Kuttler and V. G. Sigillito in [16]. The eigenvalues of (1.19) can be arranged as

$$0 = \xi_0 < \xi_1 \leq \xi_2 \leq \dots \rightarrow +\infty. \quad (1.20)$$

The first nonzero eigenvalue is given by the following Rayleigh-Ritz formula.

$$\xi_1 = \min_{\substack{0 \neq u \in H^2(\Omega) \\ \int_{\partial\Omega} u = 0, \partial_\nu u|_{\partial\Omega} = 0}} \frac{\int_{\Omega} (\overline{\Delta} u)^2}{\int_{\partial\Omega} u^2}. \quad (1.21)$$

When $\Omega = B$ (the unit ball in \mathbf{R}^n) we shall determine explicitly all the eigenvalues of (1.19). For each $k = 0, 1, \dots$, let \mathcal{D}_k be the space of harmonic homogeneous polynomials in \mathbf{R}^n of degree k and denote by μ_k the dimension of \mathcal{D}_k . We refer to [2] for the basic properties of \mathcal{D}_k and μ_k . In particular, we have

$$\begin{aligned} \mathcal{D}_0 &= \text{span}\{1\}, \quad \mu_0 = 1, \\ \mathcal{D}_1 &= \text{span}\{x_i, i = 1, \dots, n\}, \quad \mu_1 = n, \\ \mathcal{D}_2 &= \text{span}\{x_i x_j, x_1^2 - x_h^2, 1 \leq i < j \leq n, h = 2, \dots, n\}, \quad \mu_2 = \frac{n^2 + n - 2}{2}. \end{aligned}$$

Theorem 1.5. *If $n \geq 2$ and $\Omega = B$, then we have*

- i) *the eigenvalues of (1.19) are $\xi_k = k^2(n + 2k)$, $k = 0, 1, 2, \dots$;*
- ii) *for all $k = 0, 1, 2, \dots$, the multiplicity of ξ_k is μ_k ;*
- iii) *for all $k = 0, 1, 2, \dots$, and all $\phi_k \in \mathcal{D}_k$, the function $\psi_k(x) := -2\phi_k + k(|x|^2 - 1)\phi_k(x)$ is an eigenfunction corresponding to ξ_k .*

We have an isoperimetric upper bound for the first eigenvalue of the problem (1.19).

Theorem 1.6. *Let Ω be a bounded domain with smooth boundary in \mathbf{R}^n . Then the first nonzero eigenvalue of the problem (1.19) satisfies*

$$\xi_1 \leq \frac{(n+2)|\partial\Omega|}{n|\Omega| \left(\frac{|\Omega|}{\omega_n}\right)^{2/n}}, \quad (1.22)$$

with equality holding if and only if Ω is a ball.

The method in proving Theorem 1.5 can be used to study some other kind of Steklov problems. Let us consider for example the following Steklov problem

$$\begin{cases} \overline{\Delta}^2 u = 0 & \text{in } \Omega, \\ \partial_\nu u = \partial_\nu(\overline{\Delta} u) - \zeta \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.23)$$

The eigenvalues of this problem can be arranged as

$$0 = \zeta_0 < \zeta_1 \leq \zeta_2 \leq \dots \rightarrow \infty. \quad (1.24)$$

The smallest eigenvalue of (1.23) is zero with constant eigenfunction. The Rayleigh-Ritz characterization for the first nonzero eigenvalue of (1.23) can be written as

$$\zeta_1 = \min_{\substack{u \in H^2(\Omega), u \neq \text{const.} \\ \partial_\nu u|_{\partial\Omega} = 0, \int_{\partial\Omega} u = 0}} \frac{\int_\Omega (\overline{\Delta} u)^2}{\int_{\partial\Omega} |\nabla u|^2}. \quad (1.25)$$

When Ω is a ball, we can explicitly describe the eigenvalues and the corresponding eigenfunctions to the problem (1.23). That is, we have

Theorem 1.7. *If $n \geq 2$ and $\Omega = B$, then we have*

- i) the eigenvalues of (1.23) are $\zeta_0 = 0, \zeta_k = \frac{2k^2 + nk}{k+n-2}, k = 1, 2, \dots$;*
- ii) for all $k = 0, 1, 2, \dots$, the multiplicity of ζ_k is μ_k ;*
- iii) for all $k = 0, 1, 2, \dots$, and all $\phi_k \in \mathcal{D}_k$, the function $\psi_k(x) := -2\phi_k + k(|x|^2 - 1)\phi_k(x)$ is an eigenfunction corresponding to ζ_k .*

The following isoperimetric upper bound for ζ_1 is an immediate consequence of (1.21), (1.25) and Theorem 1.6.

Theorem 1.8. *Let Ω be a bounded domain with smooth boundary in \mathbf{R}^n and η_1 the first nonzero eigenvalue of the Laplacian acting on functions on $\partial\Omega$. Then the first nonzero eigenvalue of the problem (1.23) satisfies*

$$\zeta_1 \leq \frac{1}{\eta_1} \cdot \frac{(n+2)|\partial\Omega|}{n|\Omega| \left(\frac{|\Omega|}{\omega_n} \right)^{2/n}}, \quad (1.26)$$

with equality holding if and only if Ω is a ball.

An important issue in geometric analysis is to give good estimates for the eigenvalues of various eigenvalue problems. We refer to [4-12, 14-21, 23, 26, 27, 29] and the references therein for some interesting results about the Steklov eigenvalues.

2 Proofs of Theorems 1.1-1.4

Let us first fix some notation. Let M be n -dimensional compact manifold M with boundary ∂M . We write $\langle \cdot, \cdot \rangle$ the Riemannian metric on M as well as

that induced on ∂M . Let $\overline{\nabla}$ and $\overline{\Delta}$ be the connection and the Laplacian on M , respectively. Let ν be the unit outward normal vector of ∂M . The shape operator of ∂M is given by $A(X) = \nabla_X \nu$ and the second fundamental form of ∂M is defined as $\sigma(X, Y) = \langle A(X), Y \rangle$, here $X, Y \in T\partial M$. The eigenvalues of A are called the principal curvatures of ∂M and the mean curvature H of ∂M is given by $H = \frac{1}{n-1} \text{trace } A$. For a smooth function f defined on an n -dimensional compact manifold M with boundary ∂M , the following identity holds if $h = \partial_\nu f|_{\partial M}$, $z = f|_{\partial M}$ and Ric denotes the Ricci tensor of M (see [22], p. 46):

$$\begin{aligned} & \int_M \left((\overline{\Delta} f)^2 - |\overline{\nabla}^2 f|^2 - \text{Ric}(\overline{\nabla} f, \overline{\nabla} f) \right) \\ &= \int_{\partial M} ((n-1)Hh + 2\Delta z)h + \sigma(\nabla z, \nabla z). \end{aligned} \quad (2.1)$$

Here Δ and ∇ represent the Laplacian and the gradient on ∂M with respect to the induced metric on ∂M , respectively.

We shall need the following result.

Lemma 2.1 ([28]). *Let $(M, \langle \cdot, \cdot \rangle)$ be an n -dimensional compact connected Riemannian manifold with non-negative Ricci curvature and boundary ∂M . Assume that the principal curvatures of ∂M are bounded below by a positive constant c . Then the first non-zero eigenvalue η_1 of the Laplacian acting on functions on ∂M satisfies $\eta_1 \geq (n-1)c^2$ with equality holding if and only if M is isometric to an n -dimensional Euclidean ball of radius $\frac{1}{c}$.*

Proof of Theorem 1.1. Let u be an eigenfunction corresponding to $\lambda_{1,\beta}$: Let z be an eigenfunction of ∂M corresponding to η_1 , that is, $\Delta z + \eta_1 z = 0$. Let f be the solution of the following equation

$$\begin{cases} \overline{\Delta} f = 0 & \text{in } M, \\ f|_{\partial M} = z. \end{cases} \quad (2.2)$$

Since $\int_{\partial M} z = 0$, we have from (1.5) that

$$\lambda_{1,\beta} \leq \frac{\int_M |\overline{\nabla} f|^2 + \beta \int_{\partial M} |\nabla z|^2}{\int_{\partial M} z^2} = \eta_1 \beta + \frac{\int_M |\overline{\nabla} f|^2}{\int_{\partial M} z^2} \quad (2.3)$$

Setting $h = \partial_\nu f|_{\partial M}$, we know from (2.2) and the divergence theorem that

$$\int_M |\overline{\nabla} f|^2 = \int_{\partial M} zh. \quad (2.4)$$

Since the Ricci curvature of M is bounded below by $-\kappa_0$ and the principal curvature of ∂M are bounded below by c , we have

$$\text{Ric}(\nabla z, \nabla z) \geq -\kappa_0 |\nabla z|^2, \quad \sigma(\nabla z, \nabla z) \geq c |\nabla z|^2, \quad H \geq c. \quad (2.5)$$

It then follows from Reilly's formula that

$$\begin{aligned}
\kappa_0 \int_{\partial M} zh &= \kappa_0 \int_M |\bar{\nabla} f|^2 \\
&\geq \int_M \left((\bar{\Delta} f)^2 - |\bar{\nabla}^2 f|^2 - \text{Ric}(\bar{\nabla} f, \bar{\nabla} f) \right) \\
&= \int_{\partial M} (2(\Delta z)h + (n-1)h^2 + \sigma(\nabla z, \nabla z)) \\
&\geq (n-1)c \int_{\partial M} h^2 - 2\eta_1 \int_{\partial M} hz + c \int_{\partial M} |\nabla z|^2 \\
&= (n-1)c \int_{\partial M} h^2 - 2\eta_1 \int_{\partial M} hz + c\eta_1 \int_{\partial M} z^2, \tag{2.6}
\end{aligned}$$

that is,

$$0 \geq (n-1)c \int_{\partial M} h^2 - (2\eta_1 + \kappa_0) \int_{\partial M} hz + c\eta_1 \int_{\partial M} z^2. \tag{2.7}$$

Thus, we have

$$\begin{aligned}
0 &\geq (n-1)c \int_{\partial M} \left(h - \frac{(2\eta_1 + \kappa_0)}{2(n-1)c} z \right)^2 + \left(c\lambda_1 - \frac{(2\eta_1 + \kappa_0)^2}{4(n-1)c} \right) \int_{\partial M} z^2 \\
&\geq \left(c\lambda_1 - \frac{(2\eta_1 + \kappa_0)^2}{4(n-1)c} \right) \int_{\partial M} z^2,
\end{aligned}$$

and so

$$(2\eta_1 + \kappa_0)^2 \geq 4(n-1)\eta_1 c^2$$

We can also get from (2.7) that

$$0 \geq (n-1)c \int_{\partial M} h^2 - (2\eta_1 + \kappa_0) \left(\int_{\partial M} h^2 \right)^{\frac{1}{2}} \left(\int_{\partial M} z^2 \right)^{\frac{1}{2}} + c\eta_1 \int_{\partial M} z^2, \tag{2.8}$$

which, implies that

$$\left(\int_{\partial M} h^2 \right)^{\frac{1}{2}} \leq \frac{2\eta_1 + \kappa_0 + \sqrt{(2\eta_1 + \kappa_0)^2 - 4(n-1)\eta_1 c^2}}{2(n-1)c} \left(\int_{\partial M} z^2 \right)^{\frac{1}{2}}.$$

Combining (2.3), (2.4) and (2.9), we obtain

$$\begin{aligned}
\lambda_{1,\beta} &\leq \eta_1 \beta + \frac{\int_{\partial M} zh}{\int_{\partial M} z^2} \\
&\leq \eta_1 \beta + \frac{\left(\int_{\partial M} h^2 \right)^{\frac{1}{2}}}{\left(\int_{\partial M} z^2 \right)^{\frac{1}{2}}} \\
&\leq \eta_1 \beta + \frac{2\eta_1 + \kappa_0 + \sqrt{(2\eta_1 + \kappa_0)^2 - 4(n-1)\eta_1 c^2}}{2(n-1)c}. \tag{2.9}
\end{aligned}$$

This proves (1.9).

If (1.9) takes equality sign, then we have

$$\left(\int_{\partial M} h^2 \right)^{\frac{1}{2}} = \frac{2\eta_1 + \kappa_0 + \sqrt{(2\eta_1 + \kappa_0)^2 - 4(n-1)\eta_1 c^2}}{2(n-1)c} \left(\int_{\partial M} z^2 \right)^{\frac{1}{2}}$$

and so the inequalities in (2.6) and (2.8) should take equality sign. We infer therefore

$$\nabla^2 f = 0, \quad H = c \quad (2.10)$$

and

$$h = \frac{2\eta_1 + \kappa_0 + \sqrt{(2\eta_1 + \kappa_0)^2 - 4(n-1)\eta_1 c^2}}{2(n-1)c} z. \quad (2.11)$$

Take a local orthonormal fields $\{e_i\}_{i=1}^{n-1}$ tangent to ∂M . We conclude from (2.10) and (2.11) that

$$\begin{aligned} 0 &= \sum_{i=1}^{n-1} \bar{\nabla}^2 f(e_i, e_i) = \Delta z + (n-1)Hh \\ &= -\eta_1 z + (n-1)c \cdot \frac{2\eta_1 + \kappa_0 + \sqrt{(2\eta_1 + \kappa_0)^2 - 4(n-1)\eta_1 c^2}}{2(n-1)c} z, \end{aligned} \quad (2.12)$$

which gives $\kappa_0 = 0$ and $\eta_1 = (n-1)c^2$. It then follows from Lemma 2.1 that M is isometric to an n -dimensional Euclidean ball of radius $\frac{1}{c}$. On the other hand, we know from (1.7) that for the n -dimensional Euclidean ball of radius $\frac{1}{c}$, the equality holds in (1.9). This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. Let x_1, \dots, x_n be the coordinate functions on \mathbf{R}^n . By choosing the coordinates origin properly, we can assume that

$$\int_{\partial\Omega} x_i = 0, \quad i = 1, \dots, n. \quad (2.13)$$

It then follows from the variational characterization (1.5) for $\lambda_{1,\beta}$ that for each fixed $i \in \{1, \dots, n\}$

$$\begin{aligned} \lambda_{1,\beta} \int_{\partial\Omega} x_i^2 &\leq \int_{\Omega} |\bar{\nabla} x_i|^2 + \beta \int_{\partial\Omega} |\nabla x_i|^2 \\ &= |\Omega| + \beta \int_{\partial\Omega} |\nabla x_i|^2 \end{aligned} \quad (2.14)$$

with equality holding if and only if $\beta \Delta x_i + \partial_\nu x_i = -\lambda_{1,\beta} x_i$ on $\partial\Omega$.

Summing over i from 1 to n , we get

$$\begin{aligned} \lambda_{1,\beta} \int_{\partial\Omega} \sum_{i=1}^n x_i^2 &\leq n|\Omega| + \beta \int_{\partial\Omega} \sum_{i=1}^n |\nabla x_i|^2 \\ &= n|\Omega| + (n-1)\beta |\partial\Omega|, \end{aligned} \quad (2.15)$$

with equality holding if and only if

$$\beta \Delta x_i + \partial_\nu x_i = -\lambda_{1,\beta} x_i, \quad \text{on } \partial\Omega, \quad \forall i \in \{1, \dots, n\}. \quad (2.16)$$

Take a ball $B(R, o)$ in \mathbf{R}^n of radius R centered at the origin so that $|B(R, o)| = |\Omega|$; then

$$R = \left(\frac{|\Omega|}{\omega_n} \right)^{1/n}.$$

By using the weighted isoperimetric inequality in [3], we have

$$\begin{aligned} \int_{\partial\Omega} \sum_{i=1}^n x_i^2 &\geq \int_{\partial B} \sum_{i=1}^n x_i^2 \\ &= |\partial B| R^2 \\ &= n |\Omega| \left(\frac{|\Omega|}{\omega_n} \right)^{1/n}. \end{aligned} \quad (2.17)$$

Substituting (2.17) into (2.15), one gets (1.11). If the equality holds in (1.11), then the inequalities (2.14) and (2.15) must take equality sign. It follows that the position vector $x = (x_1, \dots, x_n)$ when restricted on $\partial\Omega$ satisfies

$$\begin{aligned} \Delta x &:= (\Delta x_1, \dots, \Delta x_n) \\ &= -\frac{1}{\beta} (\partial_\nu x_1, \dots, \partial_\nu x_n) - \frac{\lambda_{1,\beta}}{\beta} (x_1, \dots, x_n) \\ &= -\frac{1}{\beta} \nu - \frac{\lambda_{1,\beta}}{\beta} (x_1, \dots, x_n). \end{aligned} \quad (2.18)$$

On the other hand, it is well known that

$$\Delta x = (n-1) \mathbf{H}, \quad (2.19)$$

where \mathbf{H} is the mean curvature vector of $\partial\Omega$ in \mathbf{R}^n . Combining (2.18) and (2.19), we have

$$x = -\frac{1}{\lambda_{1,\beta}} \nu - \frac{(n-1)\beta}{\lambda_{1,\beta}} \mathbf{H}, \quad \text{on } \partial\Omega. \quad (2.20)$$

Consider the function $g = |x|^2 : M \rightarrow \mathbf{R}$. It is easy to see from (2.20) that

$$Zg = 2\langle Z, x \rangle = 0, \quad \forall Z \in \mathfrak{X}(\partial\Omega).$$

Thus g is a constant function and so $\partial\Omega$ is a hypersphere. Theorem 1.2 follows.

Proof of Theorem 1.3. i) Let u be an eigenfunction corresponding to the first eigenvalue p_1 of the Steklov problem:

$$\begin{cases} \overline{\Delta} u = 0 & \text{in } M, \\ \partial_\nu u = p_1 u & \text{on } \partial M. \end{cases} \quad (2.21)$$

Setting $w = u|_{\partial M}$, $y = \partial_\nu u|_{\partial M}$, we obtain by substituting u into Reilly's formula that

$$\begin{aligned} \kappa \int_M |\bar{\nabla} u|^2 &\geq -2 \int_{\partial M} \nabla w \nabla y + (n-1)c \int_{\partial M} y^2 + c \int_{\partial M} |\nabla w|^2 \\ &> -2p_1 \int_{\partial M} |\nabla w|^2 + c \int_{\partial M} |\nabla w|^2. \end{aligned} \quad (2.22)$$

Since $\int_{\partial M} w = 0$ and $w \neq 0$, we know from the Poincaré inequality that

$$\int_{\partial M} w^2 \leq \frac{1}{\eta_1} \int_{\partial M} |\nabla w|^2.$$

Thus we have from the divergence theorem that

$$\int_M |\bar{\nabla} u|^2 = \int_M \operatorname{div}(u \bar{\nabla} u) = \int_{\partial M} w y = p_1 \int_{\partial M} w^2 \leq \frac{p_1}{\eta_1} \int_{\partial M} |\nabla w|^2. \quad (2.23)$$

Combining (2.22) with (2.23), we obtain (1.12).

ii) Let f be an eigenfunction corresponding to the first eigenvalue $\lambda_{1,\beta}$ of the Wentzell-Laplace operator of M . Setting

$$\gamma = \frac{1}{\beta}, \quad \lambda = \frac{\lambda_{1,\beta}}{\beta}, \quad z = f|_{\partial M}, \quad h = \partial_\nu f|_{\partial M},$$

we have

$$\bar{\Delta} f = 0 \quad (2.24)$$

and

$$\Delta z = \gamma h - \lambda z. \quad (2.25)$$

It then follows from Reilly's formula that

$$\begin{aligned} 0 &\geq \int_M \left((\bar{\Delta} f)^2 - |\bar{\nabla}^2 f|^2 - \operatorname{Ric}(\bar{\nabla} f, \bar{\nabla} f) \right) \\ &= \int_{\partial M} (2(\Delta z)h + (n-1)Hh^2 + \sigma(\nabla z, \nabla z)) \\ &\geq \int_{\partial M} (2(\gamma h - \lambda z)h + (n-1)ch^2 + c|\nabla z|^2) \\ &= \int_{\partial M} (2(\gamma h - \lambda z)h + (n-1)ch^2 - cz\Delta z) \\ &= \int_{\partial M} (c\lambda z^2 - (2\lambda + c\gamma)zh + (2\gamma + (n-1)c)h^2) \\ &= (2\gamma + (n-1)c) \int_{\partial M} \left(h - \frac{2\lambda + c\gamma}{2(2\gamma + (n-1)c)} z \right)^2 \\ &\quad + \left(c\lambda - \frac{(2\lambda + c\gamma)^2}{4(2\gamma + (n-1)c)} \right) \int_{\partial M} z^2 \\ &\geq \left(c\lambda - \frac{(2\lambda + c\gamma)^2}{4(2\gamma + (n-1)c)} \right) \int_{\partial M} z^2, \end{aligned} \quad (2.26)$$

which, gives

$$c\lambda - \frac{(2\lambda + c\gamma)^2}{4(2\gamma + (n-1)c)} \leq 0.$$

Thus, we have either

$$\lambda \geq \frac{(\gamma + (n-1)c + \sqrt{(n-1)^2c^2 + 2(n-1)\gamma c})c}{2}, \quad (2.27)$$

or

$$\lambda \leq \frac{(\gamma + (n-1)c - \sqrt{(n-1)^2c^2 + 2(n-1)\gamma c})c}{2}. \quad (2.28)$$

We *claim* that (2.28) does not occur. In fact, multiplying (2.25) by $-z$ and integrating on ∂M , we get by using the divergence theorem and (2.24) that

$$\begin{aligned} \lambda \int_{\partial M} z^2 &= \int_{\partial M} |\nabla z|^2 + \gamma \int_{\partial M} h z \\ &= \int_{\partial M} |\nabla z|^2 + \gamma \int_M |\bar{\nabla} f|^2. \end{aligned} \quad (2.29)$$

Sine $\int_{\partial M} z = 0$, $z \neq 0$, we have from the Poincaré inequality and the variational characterization (1.5) that

$$\int_{\partial M} |\nabla z|^2 \geq \eta_1 \int_{\partial M} z^2, \quad \int_M |\bar{\nabla} f|^2 \geq p_1 \int_{\partial M} z^2. \quad (2.30)$$

Also, we know from (1.12) that

$$p_1 > \frac{c}{2}. \quad (2.31)$$

From (2.29)-(2.31) and Lemma 2.1, we conclude that

$$\lambda > \eta_1 + \frac{c\gamma}{2} \geq (n-1)c^2 + \frac{c\gamma}{2}, \quad (2.32)$$

which proves our *claim*. In order to finish the proof of Theorem 1.2, we need only to exclude the equality case in (2.27). We shall do this by contradiction. Thus suppose that

$$\lambda = \frac{(\gamma + (n-1)c + \sqrt{(n-1)^2c^2 + 2(n-1)\gamma c})c}{2}. \quad (2.33)$$

From the proof above, we know that all the inequalities in (2.26) should take equality sign. It then follows that the principal curvatures of ∂M satisfy

$$\rho_1 = \cdots = \rho_{n-1} = c \quad \text{on } M, \quad (2.34)$$

that

$$\overline{\nabla}^2 f = 0 \quad \text{on } M \quad (2.35)$$

and that

$$h = \frac{2\lambda + c\gamma}{2(2\gamma + (n-1)c)} z \quad \text{on } \partial M. \quad (2.36)$$

Restricting (2.35) on ∂M and using (2.34), we conclude that

$$h = cz. \quad (2.37)$$

Combining (2.36) with (2.37), we know that

$$\frac{2\lambda + c\gamma}{2(2\gamma + (n-1)c)} = c. \quad (2.38)$$

We infer from (2.33) and (2.38) that $\gamma = 0$. This is a contradiction. The proof of Theorem 1.2 is complete.

Theorem 1.4 is a special case of a more general result. Before stating it, let's introduce the following

Definition 2.1. *Let M be n -dimensional compact manifold M with boundary and τ a fixed positive number. We define the τ -fundamental tone of M to be*

$$F_{\tau, M} = \min_{\substack{0 \neq u \in H^2(\Omega) \\ \int_{\partial\Omega} u = 0}} \frac{\int_{\Omega} (|\overline{\nabla}^2 u|^2 + \tau |\overline{\nabla} u|^2)}{\int_{\partial\Omega} u^2}. \quad (2.39)$$

We have the following result which implies Theorem 1.4.

Theorem 2.2. *Let M be an n -dimensional compact manifold with boundary and assume that the principle curvatures of ∂M are bounded below by a positive constant c . Let η_1 be the first non-zero eigenvalue of the Laplacian acting on functions on ∂M . If the Ricci curvature of M is bounded below by $-\kappa$ for some constant $\kappa \geq 0$, then the τ -fundamental tone of M satisfies*

$$F_{\tau, M} \leq \frac{(2\eta_1 + \kappa)^2 + 2\tau \left(2\eta_1 + \kappa + \sqrt{(2\eta_1 + \kappa)^2 - 4(n-1)\eta_1 c^2} \right)}{4(n-1)c} - c\eta_1, \quad (2.40)$$

with equality holding if and only if $\kappa = 0$ and M is isometric to a ball of radius $\frac{1}{c}$ in \mathbf{R}^n .

Proof of Theorem 2.2. As in the proof of Theorem 1.1, let z be an eigenfunction of ∂M corresponding to η_1 and f the solution of the equation

$$\begin{cases} \overline{\Delta} f = 0 & \text{in } M, \\ f|_{\partial M} = z. \end{cases} \quad (2.41)$$

Setting $h = \partial_\nu f|_{\partial M}$, we have from the definition of $F_{\tau, M}$ that

$$\begin{aligned} F(\tau, M) &\leq \frac{\int_M |\bar{\nabla}^2 f|^2 + \tau \int_M |\bar{\nabla} f|^2}{\int_{\partial M} z^2} \\ &= \frac{\int_M |\bar{\nabla}^2 f|^2 + \tau \int_M zh}{\int_{\partial M} z^2}. \end{aligned} \quad (2.42)$$

Since

$$\text{Ric}(\bar{\nabla} f, \bar{\nabla} f) \geq -\kappa |\bar{\nabla} f|^2, \quad H \geq c, \quad \sigma(\nabla z, \nabla z) \geq c |\nabla z|^2, \quad (2.43)$$

we know from Reilly's formula that

$$\begin{aligned} - \int_M |\bar{\nabla}^2 f|^2 + \int_M \kappa |\bar{\nabla} f|^2 &\geq (n-1)c \int_{\partial M} h^2 - 2\eta_1 \int_{\partial M} hz + c \int_{\partial M} |\bar{\nabla} z|^2 \\ &= (n-1)c \int_{\partial M} h^2 - 2\eta_1 \int_{\partial M} hz - c\eta_1 \int_{\partial M} z^2, \end{aligned}$$

which gives

$$\begin{aligned} \frac{\int_M |\bar{\nabla}^2 f|^2}{\int_{\partial M} z^2} &\leq -(n-1)c \frac{\int_{\partial M} h^2}{\int_{\partial M} z^2} + (2\eta_1 + \kappa) \left(\frac{\int_{\partial M} h^2}{\int_{\partial M} z^2} \right)^{1/2} - c\eta_1 \\ &\leq -c\eta_1 + \frac{(2\eta_1 + \kappa)^2}{4(n-1)c}. \end{aligned} \quad (2.44)$$

Also, as in the proof of Theorem 1.1, we have (Cf. (2.9)), we have

$$\frac{\int_M zh}{\int_{\partial M} z^2} \leq \frac{2\eta_1 + \kappa + \sqrt{(2\eta_1 + \kappa)^2 - 4(n-1)\eta_1 c^2}}{2(n-1)c}. \quad (2.45)$$

Substituting (2.44) and (2.45) into (2.42), one gets (2.40).

If the equality holds in (2.40), we can use the same arguments as in the proof of Theorem 1.1 to conclude that $\eta_1 = (n-1)c^2$. Thus, M is isometric to a ball of radius $\frac{1}{c}$ by Lemma 2.1. Also, if M is a ball of radius $\frac{1}{c}$ in \mathbf{R}^n , we have $\eta_1 = (n-1)c^2$, $\text{Ric}_M = 0$, $F_{\tau, M} = \frac{\tau}{c}$ and therefore, the equality holds in (2.40). This completes the proof of Theorem 2.2.

3 Proofs of Theorems 1.5-1.8

In this section, we shall prove Theorems 1.5-1.7.

Proof of Theorem 1.5. Let u be an eigenfunction of (1.19) with $\Omega = B$ corresponding to an eigenvalue ξ . Since u is bi-harmonic, there exist uniquely determined harmonic functions g, h on B , such that $u(x) = g(x) + |x|^2 h(x)$ (Cf. [1]). The Euler operator Λ will play a key role in the proof :

$$\Lambda f(x) := \sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}(x).$$

We will use the notation $\Lambda^l f = \Lambda(\Lambda^{l-1} f)$, $l = 1, 2, \dots$. Observe that Λ is related to the exterior normal derivative, $\partial_\nu f|_{\partial B} = \Lambda f|_{\partial B}$, and if f is harmonic then so is Λf . Setting $g + h = -2z$, we have $u(x) = -2z(x) + (|x|^2 - 1)h(x)$. From

$$0 = \partial_\nu u|_{\partial B} = (-2\partial_\nu z + 2h)|_{\partial B} = (-2\Lambda z + 2h)|_{\partial B}$$

and the fact $-2\Lambda z + 2h$ is harmonic, we know that $-2\Lambda z + 2h = 0$ on B and so we have

$$u(x) = -2z(x) + (|x|^2 - 1)\Lambda z(x).$$

It is easy to see that

$$\overline{\Delta}u = 2n\Lambda z + 4\Lambda^2 z.$$

Thus

$$\partial_\nu(\overline{\Delta}u)|_{\partial B} = (2n\Lambda^2 z + 4\Lambda^3 z)|_{\partial B} = -\xi u|_{\partial B} = 2\xi z|_{\partial B}, \quad (3.1)$$

which gives the following important relation:

$$n\Lambda^2 z + 2\Lambda^3 z = \xi z \quad \text{on } B. \quad (3.2)$$

Since z is harmonic on B , there exist $p_m \in \mathcal{D}_m$ such that

$$z(x) = \sum_{m=0}^{\infty} p_m(x) \quad (3.3)$$

for all $x \in B$, the series converging absolutely and uniformly on compact subsets of B (Cf. [2]). Substituting (3.3) into (3.2) and observing $\Lambda p_m = m p_m$, we get

$$2m^3 + nm^2 = \xi, m = 0, 1, \dots \quad (3.4)$$

This implies that all but one of the p'_m s are zeros. Thus the eigenvalues are $\xi_k = 2k^3 + nk^2$, $k = 0, 1, \dots$, and the multiplicity of ξ_k is the dimension of \mathcal{D}_k . Also, we know from the above proof that for all $\phi_k \in \mathcal{D}_k$, the function $\psi_k(x) := -2\phi_k + k(|x|^2 - 1)\phi_k(x)$ is an eigenfunction corresponding to ξ_k . This completes the proof of Theorem 1.5.

Proof of Theorem 1.6. Let $x_i, i = 1, \dots, n$, be the coordinate functions on \mathbf{R}^n . By taking the coordinates origin properly, we can assume that

$$\int_{\Omega} x_i = 0, \quad i = 1, \dots, n. \quad (3.5)$$

For each $i \in \{1, \dots, n\}$, let g_i be the solution of the problem

$$\begin{cases} \overline{\Delta}g_i = x_i & \text{in } \Omega, \\ \partial_\nu g_i|_{\partial\Omega} = 0. \end{cases} \quad (3.6)$$

We can assume without loss of generality that

$$\int_{\partial\Omega} g_i = 0, \quad i = 1, \dots, n. \quad (3.7)$$

By using the Rayleigh-Ritz characterization (1.21), we get

$$\xi_1 \leq \frac{\int_{\Omega} x_i^2}{\int_{\partial\Omega} g_i^2}, \quad i = 1, \dots, n. \quad (3.8)$$

By using (3.6) and the divergence theorem we get

$$\int_{\Omega} x_i^2 = \int_{\Omega} x_i \overline{\Delta} g_i = - \int_{\Omega} \langle \overline{\nabla} x_i, \overline{\nabla} g_i \rangle = - \int_{\partial\Omega} g_i \partial_{\nu} x_i, \quad (3.9)$$

which implies that

$$\left(\int_{\Omega} x_i^2 \right)^2 \leq \int_{\partial\Omega} (\partial_{\nu} x_i)^2 \int_{\partial\Omega} g_i^2. \quad (3.10)$$

Substituting (3.10) into (3.8), we infer

$$\xi_1 \leq \frac{\int_{\partial\Omega} (\partial_{\nu} x_i)^2}{\int_{\Omega} x_i^2}, \quad i = 1, \dots, n. \quad (3.11)$$

Summing over i , we have

$$\xi_1 \int_{\Omega} \rho^2 \leq \int_{\partial\Omega} \sum_{i=1}^n (\partial_{\nu} x_i)^2 = |\partial\Omega|, \quad (3.12)$$

where $\rho = d(o, \cdot) : \Omega \rightarrow \mathbf{R}$ is the distance function from the origin. Let $B(R, o)$ be the ball of radius R centered at the origin in \mathbf{R}^n such that $|B(R, o)| = |\Omega|$ and set $W = \Omega \cap B(R, o)$; then

$$\begin{aligned} \int_{\Omega} \rho^2 &= \int_W \rho^2 + \int_{\Omega \setminus W} \rho^2 \\ &\geq \int_W \rho^2 + \int_{\Omega \setminus W} R^2 \\ &= \int_W \rho^2 + R^2 |B(R, o) \setminus W| \\ &\geq \int_W \rho^2 + \int_{B(R, o) \setminus W} \rho^2 \\ &= \int_{B(R, o)} \rho^2 \\ &= \frac{n\omega_n}{n+2} R^{n+2} \\ &= \frac{n\omega_n}{n+2} \left(\frac{|\Omega|}{\omega_n} \right)^{\frac{n+2}{n}}, \end{aligned} \quad (3.13)$$

which, combining with (3.12), gives (1.22). If the equality holds in (1.22), then the inequalities (3.10)-(3.13) should take equality sign, which easily implies that Ω is a ball. On the other hand, we know by scaling and Theorem 1.5 that

$$\xi_1(B(R, o)) = \frac{n+2}{R^3} = \frac{(n+2)|\partial B(R, o)|}{n|B(R, o)| \left(\frac{|B(R, o)|}{\omega_n} \right)^{2/n}}, \quad (3.14)$$

that is, the equality holds for balls in (1.22). This completes the proof of Theorem 1.6.

Proof of Theorem 1.7. Let f be an eigenfunction of (1.23) corresponding to a nonzero eigenvalue ζ . Since f is bi-harmonic with $\partial_\nu f|_{\partial B} = 0$, by using the same arguments as in the proof of Theorem 1.5, there exists a uniquely determined harmonic function w on B , such that

$$f(x) = -2w(x) + (|x|^2 - 1)\Lambda w(x).$$

We have

$$\overline{\Delta}f = 2n\Lambda w + 4\Lambda^2 w \quad (3.15)$$

and if $y \in \partial B$,

$$\begin{aligned} \overline{\Delta}f(y) &= \Delta f(y) + \overline{\nabla}^2(\nu, \nu)(y) \\ &= \Delta f(y) + (\Lambda^2 f - \Lambda f)(y) \\ &= \Delta f(y) + \Lambda^2 f(y). \end{aligned} \quad (3.16)$$

Also, we have

$$\partial_\nu(\overline{\Delta}f)|_{\partial B} = (2n\Lambda^2 w + 4\Lambda^3 w)|_{\partial B} \quad (3.17)$$

It is easy to see that for any $x \in B$,

$$\begin{aligned} \Lambda f(x) &= -2\Lambda w(x) + \sum_{i=1}^n x_i((|x|^2 - 1)\Lambda w)_{x_i} \\ &= (|x|^2 - 1)(2\Lambda w + \Lambda^2 w)(x) \end{aligned} \quad (3.18)$$

and

$$\Lambda^2 f(x) = 2|x|^2(2\Lambda w + \Lambda^2 w)(x) + (|x|^2 - 1)(2\Lambda^2 w + \Lambda^3 w)(x).$$

Hence

$$\Lambda^2 f(y) = 2(2\Lambda w + \Lambda^2 w)(y), \quad \forall y \in \partial B, \quad (3.19)$$

which, combining with (3.15) and (3.16), gives

$$\Delta f(y) = ((2n - 4)\Lambda w + 2\Lambda^2 w)(y), \quad \forall y \in \partial B. \quad (3.20)$$

It then follows from $(\partial_\nu(\overline{\Delta}f) - \zeta\Delta f)|_{\partial B} = 0$, (3.17) and (3.20) that

$$(2\Lambda^3 w + n\Lambda^2 w - \zeta(\Lambda^2 w + (n-2)\Lambda w))|_{\partial B} = 0. \quad (3.21)$$

Consequently, we have

$$2\Lambda^3 w + n\Lambda^2 w = \zeta(\Lambda^2 w + (n-2)\Lambda w) \quad \text{on } B. \quad (3.22)$$

Using the same arguments as in the final part of the proof of Theorem 1.5, we can conclude the conclusions of Theorem 1.7.

Proof of Theorem 1.8. Let u be an eigenfunction corresponding to the first eigenvalue ξ_1 of the problem:

$$\begin{cases} \overline{\Delta}^2 u = 0 & \text{in } \Omega, \\ \partial_\nu u = \partial_\nu(\overline{\Delta}u) + \xi_1 u = 0 & \text{on } \partial\Omega; \end{cases} \quad (3.23)$$

then

$$\xi_1 = \frac{\int_\Omega (\overline{\Delta}u)^2}{\int_{\partial\Omega} u^2}. \quad (3.24)$$

It follows from the variational characterization (1.25) for ζ_1 that

$$\zeta_1 \leq \frac{\int_\Omega (\overline{\Delta}u)^2}{\int_{\partial\Omega} |\nabla u|^2}. \quad (3.25)$$

We have from the Poincaré inequality that

$$\int_{\partial\Omega} |\nabla u|^2 \geq \eta_1 \int_{\partial\Omega} u^2, \quad (3.26)$$

with equality holding if and only if $u|_{\partial\Omega}$ is an eigenfunction corresponding to the eigenvalue η_1 . Combining (3.24)-(3.26) with Theorem 1.6, one can finish the proof of Theorem 1.8.

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